Stat 6341.001: Numerical Linear Algebra and Statistical Computing, Fall 2026 **Topic 1**

Closed Functions and Subgradient

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1 Introduction

Recall : Norms are different measures of how large a vector or matrix is.
On $\mathbb{R}^n$:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1, \qquad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i| .$$

(Note: $p = 2 \Rightarrow \|\mathbf{x}\|_2$ is the Euclidean norm.)

On $\mathbb{R}^{m\times n}$:

$$\|A\|_2 = \sigma_{\max}(A) \qquad \text{(largest singular value)}$$

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |A_{ij}| \qquad \text{(max column sum)}$$

$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |A_{ij}| \qquad \text{(max row sum)}$$

2 The Dual Space

Let E be a vector space. Its *dual space* is

$$E^* = \{\text{all linear functionals on } E\},$$

where a linear functional is a linear transformation from E to $\mathbb{R}$.

Lemma 2.1. *For an inner product space E, for any $f \in E^*$ there exists a unique $\mathbf{v} \in E$ such that*

$$f(\mathbf{x}) = \langle \mathbf{v}, \mathbf{x} \rangle.$$

That is, for every linear functional we can find a corresponding element of E, so $E^* = E$.

Definition 2.2 (Dual norm)**.** Given a norm $\|\cdot\|$ on E, the dual norm on E^* is

$$\|\mathbf{y}\|_* := \underbrace{\sup_{\|\mathbf{x}\|\leq 1} \langle \mathbf{y}, \mathbf{x} \rangle}_{(\text{all } \mathbf{x} \text{ in unit ball})} = \underbrace{\sup_{\|\mathbf{x}\|=1} \langle \mathbf{y}, \mathbf{x} \rangle}_{(\text{unit sphere})}$$

The two suprema will be the same.

Lemma 2.3. *Consider $p, q \geq 1$, satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then on $\mathbb{R}^n$, the dual norm of the p-norm is the q-norm.*

Example 2.4.

$$\|\cdot\|_1 \overset{\text{dual norm}}{\longleftrightarrow} \|\cdot\|_\infty, \qquad \|\cdot\|_2 \overset{\text{dual norm}}{\longleftrightarrow} \|\cdot\|_2.$$

The dual norm of the Euclidean norm is itself: if E has $\|\cdot\|_2$, then E^* has $\|\cdot\|_2$.

Lemma 2.5 (Generalized Cauchy-Schwarz inequality)**.**

$$|\langle \mathbf{y}, \mathbf{x}\rangle| \le \|\mathbf{y}\|_* \cdot \|\mathbf{x}\|, \qquad \forall \mathbf{y} \in E^*,\ \mathbf{x} \in E.$$

Remark 2.6. If $\|\cdot\| = \|\cdot\|_2$, this is the Cauchy-Schwarz inequality

$$|\langle \mathbf{y}, \mathbf{x}\rangle| \le \|\mathbf{y}\|_2 \cdot \|\mathbf{x}\|_2.$$

3 Extended Real-Valued Functions and Closedness

Our goal is to minimize some function, where E is the domain. We consider extended real-valued functions

$$f : E \to [-\infty, \infty],$$

where we allow the value ∞ in the range.

Definition 3.1 (Effective domain)**.**

$$\mathrm{dom}(f) := \{\mathbf{x} \in E : f(\mathbf{x}) < \infty\}.$$

Since we want to minimize, we do not allow the function to reach ∞ on its domain.

Example 3.2 (Indicator function)**.** Consider $C \subseteq E$. Then the indicator function of C is

$$\delta_C(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} \in C \\ \infty, & \mathbf{x} \notin C, \end{cases} \qquad \mathrm{dom}(\delta_C) = C.$$

This turns a constrained problem into an unconstrained one:

$$\begin{array}{l} \min\ f(\mathbf{x}) \\ \text{s.t. } \mathbf{x} \in C \end{array} \iff \min_{\mathbf{x} \in E}\ f(\mathbf{x}) + \delta_C(\mathbf{x}),$$

where the right-hand side is an unconstrained minimization problem. Both find the smallest value of $f(\mathbf{x})$ over $\mathbf{x} \in C$.

Definition 3.3 (Epigraph)**.**

$$\mathrm{epi}(f) = \{(\mathbf{x}, y)\ :\ f(\mathbf{x}) \le y,\ \mathbf{x} \in E,\ y \in \mathbb{R}\}.$$

The epigraph is the region that lies on or above the graph of $f(\mathbf{x})$.

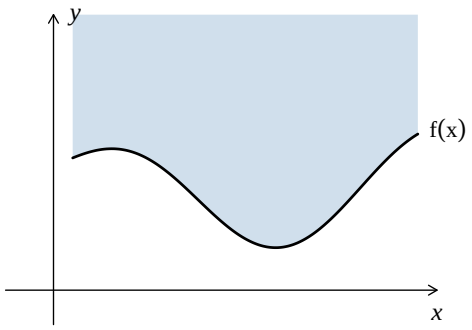


Figure 1: The epigraph of f is the region on or above the graph of f.

Definition 3.4 (Proper function)**.** f is called *proper* if there exists at least one $\mathbf{x} \in E$ such that $f(\mathbf{x}) < \infty$, i.e., $\operatorname{dom}(f) \neq \emptyset$. As long as the function is not $+\infty$ for every value, it is called proper.

Definition 3.5 (Closed function)**.** A function $f : E \to [-\infty, \infty]$ is *closed* if its epigraph is closed.

Example 3.6.

$$f(x) = \begin{cases} 0, & 1 < x < 2 \\ \infty, & \text{otherwise} \end{cases} = \delta_{(1,2)}(x).$$

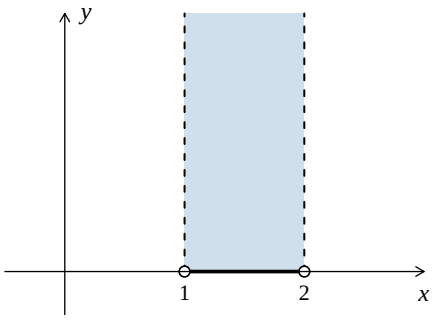


Figure 2: The epigraph of $\delta_{(1,2)}$. The dashed boundary lines are not included, so the epigraph is not closed.

Proposition 3.7. *The indicator function δ_C is closed if and only if C is a closed set.*

Proof.

$$\operatorname{epi}(\delta_C) = \{(\mathbf{x}, y) \ : \ \delta_C(\mathbf{x}) \leq y, \ \mathbf{x} \in E, \ y \in \mathbb{R}\} = C \times \mathbb{R}_+,$$

since $\mathbf{x}$ can take any value in C and y any nonnegative real value. Hence $\operatorname{epi}(\delta_C)$ is closed if and only if C is closed. □

Example 3.8. Let $f : \mathbb{R} \to [-\infty, \infty]$ be given by

$$f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \infty, & \text{else} \end{cases}.$$

Then $\operatorname{epi}(f) = \left\{(x, y) \ : \ x > 0, \ \frac{1}{x} \leq y\right\}$ is closed, so f is a closed function. Note that $\operatorname{dom}(f) = (0, \infty)$ is open, so a closed function need not have a closed domain.

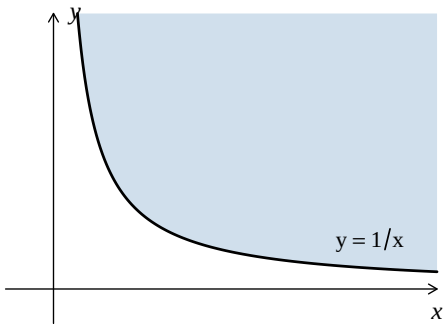


Figure 3: The epigraph of $f(x) = 1/x$.

Definition 3.9 (Lower semicontinuity)**.** A function $f : E \to [-\infty, \infty]$ is called *lower semicontinuous at* $\mathbf{x} \in E$ if

$$f(\mathbf{x}) \leq \liminf_{n\to\infty} f(\mathbf{x}_n)$$

for any sequence $\{\mathbf{x}_n\} \subset E$ with $\mathbf{x}_n \to \mathbf{x}$. It is *lower semicontinuous* if it is lower semicontinuous at every $\mathbf{x} \in E$.

Definition 3.10 (Upper semicontinuity)**.** A function $f : E \to [-\infty, \infty]$ is called *upper semicontinuous at* $\mathbf{x} \in E$ if

$$f(\mathbf{x}) \geq \liminf_{n\to\infty} f(\mathbf{x}_n)$$

for any sequence $\{\mathbf{x}_n\} \subset E$ with $\mathbf{x}_n \to \mathbf{x}$. It is *upper semicontinuous* if it is upper semicontinuous at every $\mathbf{x} \in E$.

Remark 3.11. A function is continuous if and only if it is both lower and upper semicontinuous.

Definition 3.12 (α-level set)**.**

$$\mathrm{Lev}(f, \alpha) = \{\mathbf{x} \in E \ : \ f(\mathbf{x}) \leq \alpha\}.$$

Theorem 3.13. *Let $f : E \to [-\infty, \infty]$. Then the following three statements are equivalent:*

(i) f is lower semicontinuous.

(ii) f is closed.

(iii) For any $\alpha \in \mathbb{R}$, $\mathrm{Lev}(f, \alpha)$ is closed.

Remark 3.14. Any continuous function is lower semicontinuous, so it is closed.

Theorem 3.15 (Operations preserving closedness)**.**

(a) Let $\{f_i\}_{i=1}^m$, $f_i : E \to (-\infty, \infty]$, be closed, and consider $\{\alpha_i\}_{i=1}^m \subset \mathbb{R}_+$. Then $f = \sum_{i=1}^m \alpha_i f_i$ is closed.

(b) Let $\{f_i\}_{i\in I}$, $f_i : E \to (-\infty, \infty]$, be closed. Then $f(\mathbf{x}) = \max_{i\in I} f_i(\mathbf{x})$ is closed.

(c) Let f be closed. Then $g(\mathbf{x}) = f(A\mathbf{x} + \mathbf{b})$ is closed.

Remark 3.16. If f_1, f_2 are closed, then $\mathrm{epi}(\max\{f_1, f_2\}) = \mathrm{epi}(f_1) \cap \mathrm{epi}(f_2)$ is an intersection of closed sets, so it is closed.

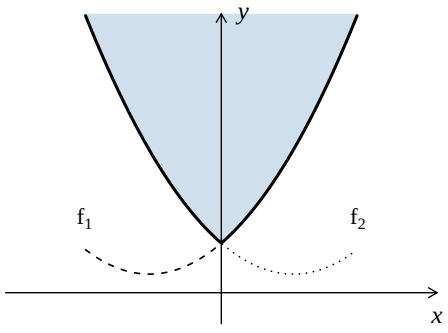


Figure 4: The epigraph of $\max\{f_1, f_2\}$.

4 Subgradient

Definition 4.1 (Subgradient)**.** Let $f : E \to (-\infty, \infty]$ be proper, and $\mathbf{x} \in \mathrm{dom}(f)$. A vector $\mathbf{g} \in E^*$ is called a *subgradient* of f at $\mathbf{x}$ if

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle, \qquad \forall \mathbf{y} \in E.$$

Remark 4.2. Recall: if f is convex and differentiable, then

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \qquad \forall \mathbf{y}.$$

Example 4.3. Let $f(x) = |x|$, $x \in \mathbb{R}$. Then f is not differentiable at $x = 0$. At $x = 0$, g is a subgradient if $\forall y$,

$$\begin{aligned} f(y) &\geq f(0) + g \cdot (y - 0) \\ &= 0 + g \cdot y \end{aligned}$$

so

$$|y| \geq g \cdot y, \qquad \forall y.$$

So we want to find some g such that this is true.

For example: $g = 0$, $g = 1$, and in general when $g \in [-1, 1]$.

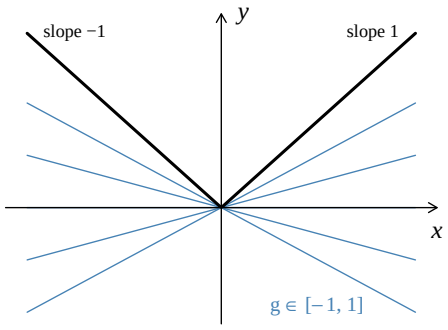


Figure 5: For all the possible slopes $g \in [-1, 1]$, the lines stay below $|x|$, so g is a subgradient of $|x|$ at 0.

References

[1] A. Beck, *First-Order Methods in Optimization*, SIAM, 2017. Sections 1.11, 2.1, 3.1